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An n-gonal quasilattice in two dimensions as a dual lattice to a multiple periodic grid

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Abstract. An n-gonal quasilattice in two dimensions is constructed for the case of $n = 6m$ $(m \geqslant 2)$ as a dual to an m-tuple periodic grid $G_m = G_0 \cup rG_0 \cup \ldots \cup r^{m-1}G_0$, where G_0 is a periodic grid with hexagonal point symmetry and r the rotation operation of the plane by angle $2\pi/n$. We investigate the case where G_0 is a honeycomb, triangular, Kagomé or diced grid. The resulting quasilattice is not of 'Bravais' type unless G_0 is the honeycomb grid. It is shown also that the same quasilattice is obtained with the projection method from the $2m$-dimensional lattice $L = L_0 \times L_0 \times \ldots \times L_0$, where L_0 is a triangular, honeycomb, diced or Kagomé lattice being dual to G_0. In the projection method, the $2m$-dimensional Euclidean space is divided into cells with a $2m$-dimensional grid G which is dual to L. It is shown that G_m is identical to a two-dimensional section of G.

1. Introduction

The most general method of constructing an n-gonal quasilattice (n being even) in two dimensions (2D) is the projection (more exactly, the cut-and-projection) method (Niizeki 1989a, b); an n-gonal quasilattice has D_n, the dihedral group of order $2n$, as its macroscopic point symmetry. In this method, we start from a periodic n-gonal lattice in a higher dimension and cut the lattice with a slab (more exactly, a hyperslab) being parallel to a 2D subspace, which is referred to as the external space. Then, an n-gonal quasilattice is obtained by projecting the cut lattice points onto the subspace. If the starting lattice is a Bravais lattice with the minimal dimension given by $\varphi(n)$, with φ being the Eulerian function in number theory, we obtain a 'Bravais-type' quasilattice. On the other hand, if the dimension of the starting lattice is higher than the minimal dimension or the starting lattice is not a Bravais lattice, we obtain a 'non-Bravais-type' quasilattice.

In the projection method, the orthogonal complement of the external space in the Euclidean space embedding the starting lattice is referred to as the internal space. The width of the slab is given by the window which is equal to the projection of the slab onto the internal space. If $n \geqslant 14$, i.e. $\varphi(n) \geqslant 6$, the dimension of the internal space is equal to or larger than four. Then, to investigate the form of the window is a problem of higher-dimensional geometry, which is difficult to investigate visually.

A complementary method of obtaining a quasilattice to the projection method is the grid (more exactly, the dual-grid) method. In this method, an n-gonal quasiperiodic network (or tiling) associated with an n-gonal quasilattice is obtained as the dual network to a quasiperiodic grid on the same plane on which the quasilattice is put. The original grid method assumes a linear grid, which is a superposition of several simple grids; a simple grid is a periodic array of parallel lines (de Bruijn 1981, Krammer and Neri 1984, see also Niizeki 1988*b*). It is subsequently extended to the case of a multiple periodic grid, which is a superposition of several equivalent periodic grids (Stampfli 1986, Niizeki 1988a, Korepin *et al* 1988); the superposition of the equivalent periodic grids gives rise to a quasiperiodic pattern with a non-crystallographic macroscopic point symmetry. The grid method is superior to the projection method in that we can restrict our consideration to the geometry on the plane.

A quasilattice obtained with the projection method is obtained, alternatively, with the grid method if the window is given by the projection of a Voronoi cell (polytope) of the starting lattice onto the internal space (Niizeki 1988a, Korepin *et al* 1988). If the starting lattice is a simple hypercubic lattice, the corresponding grid is a linear grid (de Bruijn 1981, Gähler and Rhyner 1986, Niizeki 1988b). Under a different condition, we obtain a multiple periodic grid (Niizeki 1988a, Korepin *et al* 1988). In the general case, we obtain a more complicated grid and the merit of the grid method over the projection method is lost.

In this paper, we shall investigate in further detail the case of a multiple periodic grid over Niizeki (1988a) and Korepin *et al* (1988). We shall extend the grid method so that the case of a 'non-Bravais-type' quasilattice is included. We will extend it also to the case where the grid is derived from a non-Voronoi division of the space. We shall confine our arguments to the case of an n-gonal quasilattice with n being a multiple of 6.

In § 2, we present exact definitions of the terms, 'network', 'grid' and 'duality' which are used in this paper. In § 3, we introduce several periodic 2D grids and higher-dimensional grids. We construct in § 4 an n-gonal quasilattice with the projection method from a higher-dimensional lattice. We show in § 5 that a quasilattice constructed in § 4 is obtained, alternatively, as a dual to a multiple periodic grid. We discuss related subjects in § 6.

The theory will be developed quite generally but it will be more easily comprehended if readers consider the case where the parmeter m is equal to 2.

2. Networks and grids

We shall denote by E_d the d-dimensional ($d \geqslant 2$) Euclidean space. We consider E_d to be a vector space with the Euclidean norm. Let V be a discrete set of points in E_d. Then, we shall call V a semi-uniform system if there exists an upper limit for the radius of a ball which can be accommodated in $E_d - V$. V is a countable set because it is discrete.

Let $P = \{\{x, x'\} \mid x, x' \in V \text{ and } x \neq x'\}$, i.e. the set of all possible pairs of points in a semi-uniform system V, and let B be a subset of P. Then, we shall call $N = \{V, B\}$ a network. An element in V is called a vertex and the one in B a bond. A bond, $\{x, x'\} \in B$, is represented geometrically by a segment connecting x and x', i.e. $\{(1-t)x + tx' \mid t \in [0, 1]\}$. We shall identify hereafter the bond with the segment. N is connected if, for any two vertices x and x', there exists a chain of bonds, $\{x, x_1\}, \{x_1, x_2\}, \ldots, \{x_k, x'\}$ for some integer k. A connected network is called regular if every bond can share a point with other bonds only at its two ends. A regular network has no pairs of crossing bonds. In this paper, we mean by a 'network' only a regular network.

A d-dimensional ($d \geqslant 2$) grid G is a connected closed set of points in E_d satisfying: (i) its measure is zero; (ii) $E_d - G$ decomposes into disjoint sets which are convex polytopes (polygons or polyhedrons if $d=2$ or 3, respectively) in E_d Note that the boundaries of the Voronoi partition (of the space) with respect to a semi-uniform system form a grid. We will see various grids in later sections.

Let G be a d-dimensional grid. Then, we shall refer to connected components (polytopes) in $E_d - G$ simply as cells. The set of all the cells of G is denoted by $C = C(G)$. Note that the set of vertices of all the cells, $V = V(G)$, and the set of edges of them, $B = B(G)$, form a network $N(=N(G)) = \{V, B\}$.

A d-dimensional network (or grid) is itself a d-dimensional pattern and its symmetry is represented by a space group. If the space group coincides with that of a d-dimensional lattice, the network (or grid) is called periodic. If $N = \{V, B\}$ (or G) is periodic, V (or $V(G)$) represents a d-dimensional lattice whose space group is identical to that of N (or G). In this case, we shall use symbol L (or $L(G)$) in place of V (or $V(G)$) because it is, then, a periodic lattice.

In the rest of this section, we will discuss in more detail networks and grids in two dimensions. Although a 2D grid and the network associated with it are logically different objects, we may sometimes identify the two.

A vertex of a network is called a balanced vertex if the bonds shooting from it divide a neighbourhood of the vertex into disjoint acute sectors as explained in figure 1. The number of bonds shooting from a balanced vertex is larger than two. A network is called a balanced network if all its vertices are balanced and a grid is called a balanced grid if the associated network is balanced. In this paper, we shall confine our consideration only to balanced networks and grids. Note that a 2D grid of a brick-wall pattern which is topologically isomorphous to the honeycomb grid is not a 2D grid as defined here because every vertex is not balanced.

A network $N = \{V, B\}$ is called the dual to the grid G if (i) there exists a bijection (one-to-one correspondence) Γ from V onto $C = C(G)$ and (ii) a necessary and sufficient condition for $\{x, x'\} \in B$ with $x, x' \in V$ is that the cells $\Gamma(x)$ and $\Gamma(x')$ share

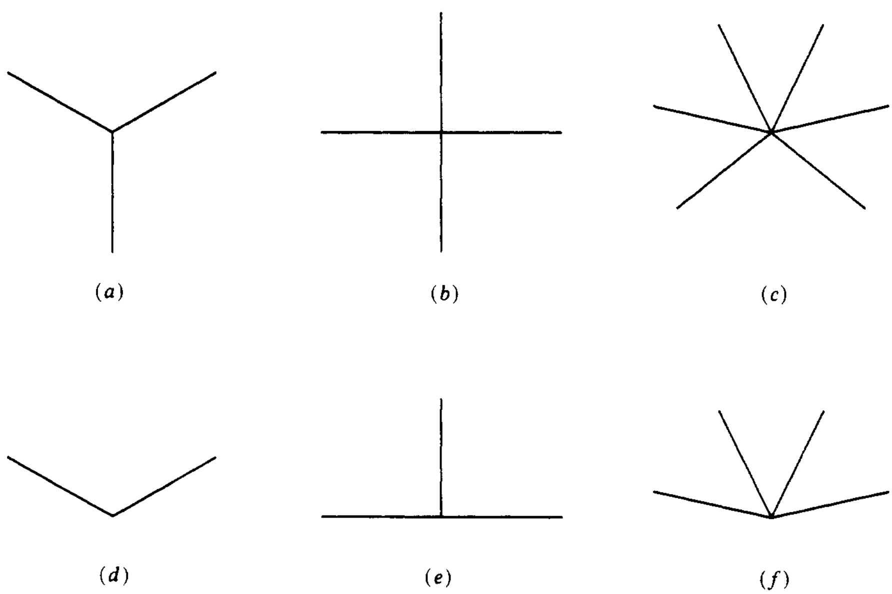


Figure 1. Vertices (a), (b) and (c) are balanced, while (d), (e) and (f) are not.

an edge. If a network $N = \{V, B\}$ is given, we can define in a reverse way a dual grid G to N.

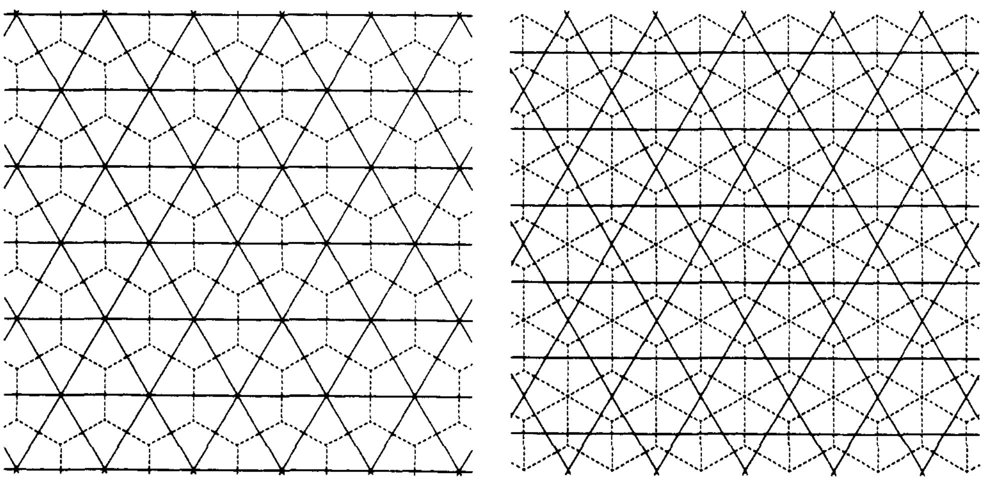
The duality is a topological relationship, so that a dual to a given grid (or a network) is not uniquely determined by Euclidean geometry. A grid (or network) always has a dual network (or grid) because we deal only with balanced networks and grids. Moreover, we may assume that the dual grid (or network) to a periodic network N (or grid G) is also periodic and the space group is common between the dual pair. In particular, if every cell in $C = C(G)$ has a centre of symmetry, the dual network $N = \{V, B\}$ is uniquely determined; V is a set of the centres of the cells in C.

3. Periodic grids in two and higher dimensions

3.1. 2D grids with hexagonal point symmetry

We shall investigate periodic 2D grids with the hexagonal point symmetry D_6. We shall identify in this subsection a grid with the network associated with it.

We assume that all the bonds of the grid (exactly network) to be considered have a common length. Moreover, we assume that a bond shooting from a vertex can take only one of the six directions which are parallel to the vertex vectors of a regular hexagon centring on the origin. The honeycomb grid, G_{HC}, the triangular grid, G_T, the Kagomé grid, G_K, and the diced grid, G_D, as given in figure 2 satisfy these conditions. G_{HC} (or G_K) and G_T (or G_D) are dual to each other. G_T and G_K are linear grids. The four grids have a common space group, i.e. P6mm. $L_T \equiv V(G_T)$ is a triangular lattice, which is a Bravais lattice, so that all the vertices are equivalent. $L_X \equiv V(G_X)$ is not a Bravais lattice for X = HC, K or D; it is obtained from L_T by decorating its unit cells appropriately. It is divided into a number of Bravais lattices



(a) (b)

Figure 2. (a) A triangular grid G_T (full lines) and a honeycomb grid G_{HC} (broken lines). They are dual to each other. Cells of G_T (or G_{HC}) are the Voronoi cells of the lattice $L_{HC} = V(G_{HC})$ (or $L_T = V(G_T)$). (b) A Kagomé grid G_K (full lines), and the diced grid G_D (broken lines). They are dual to each other. Cells of G_D are the Voronoi cells of the lattice $L_K = V(G_K)$ but those of G_K are not the Voronoi cells of L_D.

which are isomorphous to L_T; L_{HC} (or L_K) is composed of two (or three) equivalent sublattices, while L_D is composed of three sublattices, two of which are equivalent.

$C_{X'} = C(G_{X'})$ with $X' = $ HC, T, K or D is a dual set to L_X with $X = $ T, HC, D or K, respectively and, hence, a dual statement to the one given above for L_X applies to $C_{X'}$. Note also that G_{HC}, G_T or G_D is obtained with the Voronoi partition from L_T, L_{HC} or L_K, respectively (but G_K is not). It follows that a bond of one of the dual pair of G_T and G_{HC} (or G_K and G_D) is always perpendicular to the dual bond of the dual grid (see figure 2).

E_2 can be identified with C, the complex plane. Therefore, a periodic 2D grid G_X and the relevant lattice $L_X = V(G_X)$ can be considered to be subsets of C. Let $G_{X'}$ be a dual to G_X. Then, each cell in $C_X = C(G_X)$ is labelled by a lattice point in $L_{X'} = V(G_{X'})$. We shall denote the cell labelled by $z \in L_{X'}$ as $C(z)$.

3.2. *A higher-dimensional grid being dual to a network given by a direct product of 2D networks*

In this subsection, we shall identify the $2m$-dimensional ($m \geq 2$) Euclidean space E_{2m} with $C^m = C \times C \times \ldots \times C$. Moreover, we shall distinguish a 2D grid G_X with $X = $ HC, T, K or D from the relevant network N_X. Let $N_X = \{L_X, B_X\}$ ($L_X \subset C$). Then, we can construct a $2m$-dimensional periodic network $N = \{L, B\}$ as follows†. Firstly, $L = L_X \times L_X \times \ldots \times L_X \subset C^m$. Secondly, if $z = (z_0, z_1, \ldots, z_{m-1})$, $z' = (z'_0, z'_1, \ldots, z'_{m-1}) \in L$, we postulate that $\{z, z'\} \in B$ if and only if (i) $z_i = z'_i, \forall i \in \{0, 1, \ldots, k-1, k+1, \ldots, m-1\}$ for some k and (ii) $\{z_k, z'_k\} \in B_X$. We shall refer to N constructed in this way as the 'direct product' of n networks N_X and represent this fact by $N = N_X \times N_X \times \ldots \times N_X$. We may refer to $L (= L_X \times L_X \times \ldots \times L_X)$ as a hyper-X lattice (i.e. hypertriangular lattice, etc). If L_X is composed of p Bravais sublattices, L is divided into $q = p^m$ Bravais sublattices as $L = L_1 \cup L_2 \cup \ldots \cup L_q$.

Let $G_{X'}$ be the dual grid to N_X. Then, a 'dual' grid $G \subset C^m$ to $N = N_X \times N_X \times \ldots \times N_X$ is defined by $G = G^{(0)} \cup G^{(1)} \cup \ldots \cup G^{(m-1)}$, where $G^{(k)} = C^k \times G_{X'} \times C^{m-1-k}$, $k = 0, 1, \ldots, m-1$, are 2D arrays of $(2m-2)$-dimensional 'tubes' (cf Niizeki 1988a, Korepin *et al* 1988). A cell in $C = C(G)$ can be obtained quite easily; if $z = (z_0, z_1, \ldots, z_{m-1}) \in L$, the cell labelled by z is given by a convex polytope $\Gamma(z) = C(z_0) \times C(z_1) \times \ldots \times C(z_{m-1})$, which is a $2m$-dimensional hyperism (cf the fact that a Voroni cell of a simple hexagonal network $N_T \times N_1$ is a hexagonal prism). $\Gamma(z)$ is the Voronoi cell of the lattice point z in L if $X = $ HC, T or K. If $z \in L_\lambda$, the λth sublattice ($\lambda = 1, 2, \ldots, q$) of L, we may write $\Gamma(z) = z + \Gamma_\lambda$, where Γ_λ is a polytope centring at the origin. Different Γ_λ have different forms and/or orientations.

4. A construction of an *n*-gonal quasilattice with the projection method

$E_{2m} \simeq C^m$ is written as $C \oplus C \oplus \ldots \oplus C$, where the symbol $\oplus$ stands for taking a direct sum between two vector spaces with the Euclidean norm. Then, a position vector in

† This construction is a generalisation of the one by which a simple hypercubic network N_{SHC} in m dimensions is obtained as a direct product of m equivalent 1D networks N_1; $N_{SHC} = N_1 \times N_1 \times \ldots \times N_1$. Similarly, a network associated with the simple hexagonal lattice in 3D is given by $N_T \times N_1$.

Note that a network associated with a face-centred cubic lattice or diamond lattice is, for example, not represented as a direct product of lower-dimensional networks.

E_{2m} is represented by a complex vector $z=(z_0, z_1, \ldots, z_{m-1}) \in C^m$. Let us introduce an m-dimensional unitary matrix $\mathbf{R}$ which acts on $z=(z_0, z_1, \ldots z_{m-1}) \in C^m$ as $\mathbf{R}z = (\omega z_{m-1}, z_0, \ldots, z_{m-2})$ with $\omega = \exp(\pi \mathrm{i}/3) = (\sqrt{3}+\mathrm{i})/2)$. $\mathbf{R}$ represents an orthogonal transformation of $E_{2m} \simeq C^m$. We obtain that $\mathbf{R}^m = \omega I$ and $\mathbf{R}^n = \mathbf{I}$ with $n = 6m$, where $\mathbf{I}$ is the m-dimensional unit matrix. Thus, each component in $C^m = C \oplus C \oplus \ldots \oplus C$ is an invariant subspace against $\mathbf{R}^m$; $\mathbf{R}^m$ acts onto each subspace as a rotation through $\pi/6$.

Let us introduce an (anti-linear) transformation, $\mathbf{S}$, of C^m by $\mathbf{S}(z_0, z_1, \ldots, z_{m-1}) = (\bar{z}_0, \bar{z}_1, \ldots, \bar{z}_{m-1})$, where the bar stands for the complex conjugate operation. Then, we obtain $\mathbf{S}^2 = \mathbf{I}$ and $\mathbf{SRS} = \mathbf{R}^{-1}$. Therefore, $\mathbf{R}$ and $\mathbf{S}$ generate a $2m$-dimensional point symmetry group, $\tilde{\mathrm{D}}_n$, which is isomorphous to D_n.

From the secular equation, $\det(\lambda \mathbf{I} - \mathbf{R}) = \lambda^m - \omega = 0$, the eigenvalues of $\mathbf{R}$ are obtained as $\zeta_k \equiv \zeta \eta^k$, $k = 0, 1, \ldots, m-1$ with $\zeta = \exp(2\pi \mathrm{i}/n)$ and $\eta = \zeta^6 (= \exp(2\pi \mathrm{i}/m))$. The corresponding left eigenvectors are given by $u^{(k)} = (1, \zeta_k, \ldots, \zeta_k^{m-1})$, $k = 0, 1, \ldots, m-1$; $u^{(k)}\mathbf{R} = \zeta_k u^{(k)}$. The different eigenvectors are unitarily orthogonal to each other, so that C^m is written as a direct sum of m invariant 1D complex subspaces against $\mathbf{R}$. We shall denote this decomposition of C^m as $C^m = C' \oplus C' \oplus \ldots \oplus C' = (C')^m$, where $C' \simeq E'_2$ is a 1D complex vector space being isomorphous to C. Note that $\mathbf{R}$ acts on the kth subspace in $(C')^m$ as a multiplication of ζ_k.

On the other hand, $\mathbf{S}$ acts on the kth subspace in $(C')^m$ as the complex conjugate operation. Therefore, each component of $(C')^m$ is an invariant subspace against $\tilde{\mathrm{D}}_n$ which acts on the first component in $(C')^m$ as D_n.

Since N_X with $X = $ HC, T, K or D has centres of symmetry of D_6, we can assume that it is embedded in C so that $\omega N_X = N_X$ and $\bar{N}_X = N_X$. Then, the $2m$-dimensional network $N = N_X \times N_X \times \ldots \times N_X$ and the 'dual' grid G to N $(= \{L, B\})$ are invariant against $\mathbf{R}$ and $\mathbf{S}$. That is, $\tilde{\mathrm{D}}_n$ is a subgroup of the point symmetry group of N. It follows that the set of points defined by

$$L_Q = \{\pi(z) \,|\, z \in L \text{ and } \Gamma(z) \cap E'_2 \neq \phi\} \tag{1}$$

is an n-gonal quasilattice (cf Korepin *et al* 1988), where π is the projector from $(C')^m$ onto its first component $C' \simeq E'_2 = \{zv \,|\, z \in C\} \subset C^m$ with $v = (1, \bar{\zeta}, \ldots, \bar{\zeta}^{m-1})$. Note that the condition $\Gamma(z) \cap E'_2 \neq \phi$ means that the cell $\Gamma(z)$ is cut by the external space (tiling space) E'_2. Note also that, apart from a scale factor, $\pi(z)$ is equal to $u^{(0)} \cdot z = z_0 + z_1\zeta + \ldots + z_{m-1}\zeta^{m-1}$, i.e. the mapping of each subspace in C^m onto E'_2 by π is a congruent transformation.

Let $B_Q = \{\{\pi(z), \pi(z')\} \,|\, \{z, z'\} \in B \text{ and } \pi(z), \ \pi(z') \in L_Q\}$. Then, we obtain a quasiperiodic network $N_Q = \{L_Q, B_Q\}$. The vector representing the bond between $\pi(z)$ and $\pi(z')$ in B_Q is given by $\pi(z'-z)$, which is equal to ζ^k for some k because $\{z, z'\} \in B$. Thus, all the bonds in N_Q have a common length (the unit length) and the orientation of a bond is parallel to one of the vertex vectors of a regular n-gon centred on the origin.

L_Q is divided into 'Bravais sublattices' as $L_Q = L_{Q,1} \cup L_{Q,2} \cup \ldots \cup L_{Q,q}$ with $L_{Q,\lambda} = \{\pi(z) \,|\, z \in L_\lambda \text{ and } \Gamma(z) \cap E'_2 \neq \phi\}$. Let π' be the projector of C^m onto the internal space $E'_{2m-2} \simeq (C')^{m-1}$, the orthogonal complement of $C' \simeq E'_2$ in $C^m (\simeq E_{2m})$. Then, $\pi'(E'_2) = 0$, so that the condition $\Gamma(z) \cap E'_2 \neq \phi$ with $z \in L_\lambda$ is equivalent to $0 \in \pi'(z + \Gamma_\lambda)$, which is, further, equivalent to $\pi'(z) \in W_\lambda$, where $W_\lambda = -\pi'(\Gamma_\lambda)$ is the window to be assigned to sublattice L_λ. Consequently, $L_{Q,\lambda} = \{\pi(z) \,|\, z \in L_\lambda \text{ and } \pi'(z) \in W_\lambda\}$. Thus, equation (1) is a special case of the one in Niizeki (1989b) in which the phase vector is set equal to zero. Strictly speaking, L_λ in this paper is not necessarily equal to the n-gonal Bravais lattice with the minimal dimension as will be discussed in a later section.

In summary, the quasiperiodic network N_Q is the projection of a cut of a higher-dimensional periodic network N onto E_2'.

5. A construction of a quasiperiodic network (grid) as the dual to a multiple periodic grid

A lattice point in a hyper-X lattice L contributes to L_Q in equation (1) if the relevant cell of the dual grid G is cut by E_2'. Therefore, it is important to investigate the pattern of the section of G along E_2'. Since $G^{(k)} = C^k \times G_{X'} \times C^{m-1-k}$ and $E_2' = \{z\boldsymbol{v} \mid z \in C\}$, we obtain $G^{(k)} \cap E_2' = \{z\boldsymbol{v} \mid z(\bar{\zeta})^k \in G_{X'}\} = \{z\zeta^k\boldsymbol{v} \mid z \in G_{X'}\}$. It follows that $\pi(G^{(k)} \cap E_2') = \{mz\zeta^k \mid z \in G_{X'}\} \propto \zeta^k G_{X'}$, where use has been made of $\pi(\boldsymbol{v}) = m$. Since $G = G^{(0)} \cup G^{(1)} \cup \ldots \cup G^{(m-1)}$, we obtain $G_m \equiv \pi(G \cap E_2') = G_{X'} \cup \zeta G_{X'} \cup \ldots \cup \zeta^{m-1} G_{X'}$, where an unimportant scale factor, m, is omitted. Thus, the section, G_m, is found to be equal to a superposition of m equivalent periodic grids $G_{X'}, \zeta G_{X'}, \ldots, \zeta^{m-1} G_{X'}$. We shall call G_m as an m-tuple X′-grid (m-tuple honeycomb grid, etc.).

It can be readily proved that G_m is a 2D grid in the definition presented in § 2. A cell of G_m represents a 2D section of a cell of G along E_2'. Therefore, there is a bijection between $C(G_m)$ and L_Q. Moreover, the line including a common edge between two adjacent cells of G_m is a section of the boundary hypersurface Π between the relevant cells, $\Gamma(z)$ and $\Gamma(z')$ of G. The section, $\Pi \cap E_2'$, is a line being perpendicular to the direction of the bond, $\pi(z' - z)$, which follows from the fact that $z' - z$ is normal to Π. Thus, we can conclude that $N_Q = \{L_Q, B_Q\}$ is the dual network to G_m. N_Q is determined uniquely (apart from the scale) from G_m by the conditions that all the bonds of N_Q are of an equal lengths and every dual pair of the bonds are perpendicular to each other.

We have so far considered G_m as a grid and N_Q as a dual network to G_m. We may consider, conversely, G_m as a 'network' and N_Q as a dual 'grid' to G_m. In this treatment, we focus our attention rather on cells (or tiles) in N_Q than on the vertices of N_Q. Then, $V = V(G_m)$, the set of the vertices of G_m, is divided into two disjoint sets as $V = V_1 + V_2$, where $V_1 = V(G_{X'}) + V(\zeta G_{X'}) + \ldots + V(\zeta^{m-1} G_{X'})$ $(= L_{X'} + \zeta L_{X'} + \ldots + \zeta^{m-1} L_{X'})$ and V_2 is the set of all the crossing points between the bonds of $G_{X'}, \zeta G_{X'}, \ldots, \zeta^{m-1} G_{X'}$.

The dual 'grid' N_Q to the 'network' G_m divides the plane into cells. The cells are equisided polygons such as a triangle, a hexagon or a rhombus (a square is the special case of a rhombus) whose inner angles are multiples of $2\pi/n$. The lengths of the sides are common among different types of cells. A cell of a given type can assume several orientations which are consistent with the point symmetry D_n of N_Q. The dual cell to a vertex in V_2 is a rhombus because the vertex represents a crossing point between the bonds of different periodic grids in G_m. The same is true for a vertex in V_1 if $X' = $ K because it is a crossing point between two lines. On the other hand, the dual cell to a vertex in V_1 in other cases is an equisided triangle or a regular hexagon depending on whether the vertex is three-pronged or six-pronged, respectively.

It is usual that the cells of N_Q are taken to be tiles and the division of the plane into cells is taken to be a tiling. That is, N_Q represents an n-gonal quasiperiodic tiling (QPT) of the plane. A QPT obtained as the dual to a multiple triangular or Kagomé grid is a special case of a QPT obtained from a linear grid. In this paper, the m-tuple

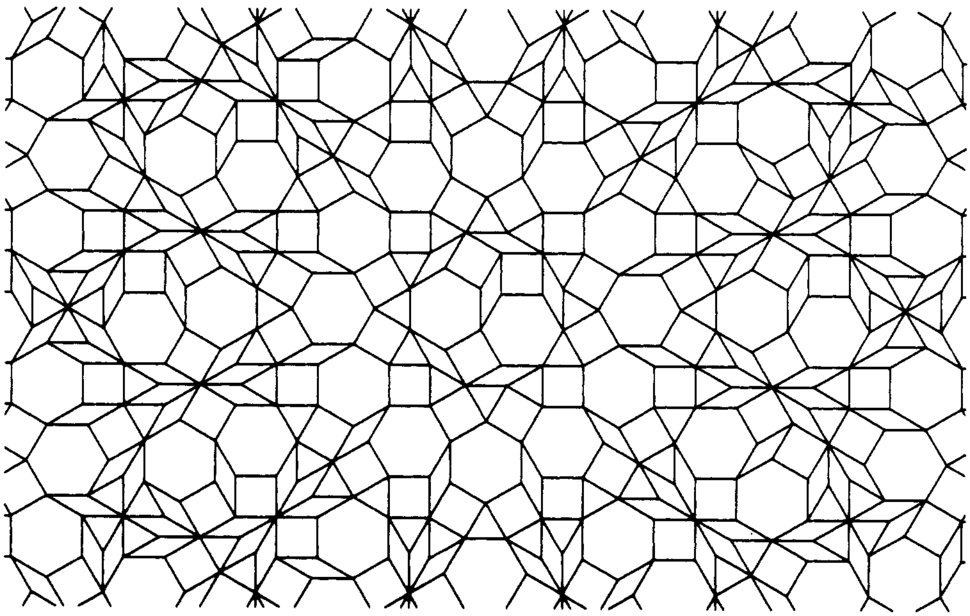


Figure 3. A dodecagonal QPT obtained as a dual to the double-diced grid as given in figure 4. It has four kinds of tiles: an equisided triangle, a regular hexagon, a square and a rhombus.

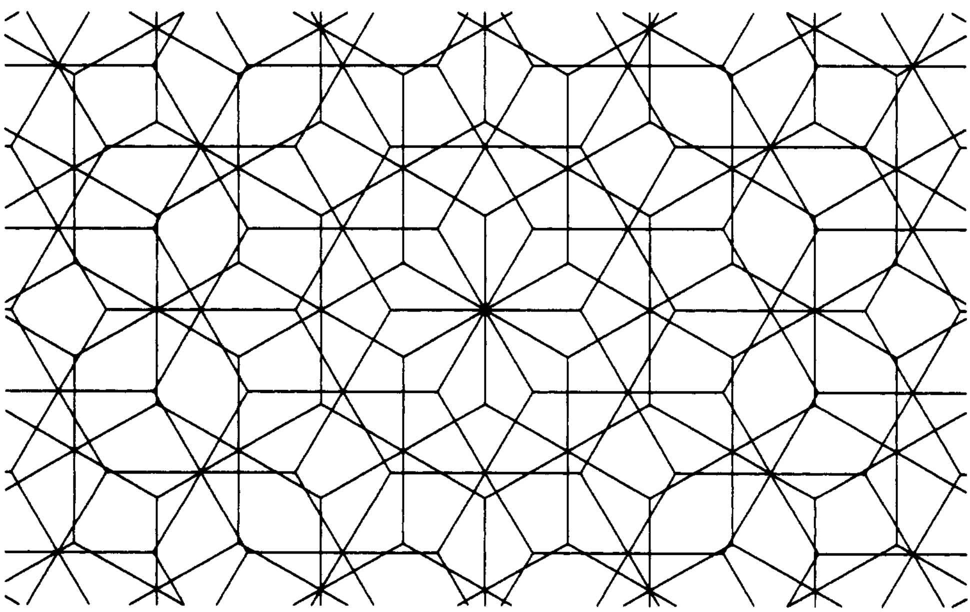


Figure 4. A double-diced grid. The 12-pronged vertex at the origin is a singular vertex caused by a coincidence of two six-pronged vertices. A six-pronged vertex of one of the two diced grids is located on a bond of the other, resulting in an eight-pronged vertex which is another kind of singular vertex.

triangular (or Kagomé) grid is derived from a higher-dimensional grid in $2m$ dimensions but it is derived, alternatively, from a simple hypercubic lattice in $3m$ dimensions (Niizeki 1988b).

A dodecagonal QPT obtained as the dual to a double honeycomb grid is presented by Stampfli (1986) and investigated by Korepin *et al* (1988). 18-gonal and 24-gonal QPT are obtained as the duals to multiple honeycomb grids by Niizeki (1989a).

A QPT obtained as the dual to a multiple-diced grid has not been presented by any author. We show in figure 3 a dodecagonal QPT obtained as the dual to a double-diced grid as given in figure 4. The tile of an equisided triangle and that of a regular hexagon are dual to a three-pronged vertex and a six-pronged vertex of the two diced grids G_{D} and ζG_{D} and a square tile (or a rhombic tile) is dual to a crossing point between two bonds which are perpendicular (or oblique) to each other.

6. Discussions

We can include the phase vector $\boldsymbol{\phi}=(\phi_0, \phi_1, \ldots, \phi_{m-1}) \in E'_{2m-2}$ into equation (1):

$$L_Q(\boldsymbol{\phi})=\{\pi(z)\,|\,z \in L \text{ and } \Gamma(z)\cap(\boldsymbol{\phi}+E'_2)\neq\phi\} \tag{2a}$$

$$= L_{Q,1}(\boldsymbol{\phi})\cup L_{Q,2}(\boldsymbol{\phi})\cup \ldots \cup L_{Q,q}(\boldsymbol{\phi}) \tag{2b}$$

$$L_{Q,\lambda}(\boldsymbol{\phi})=\{\pi(z)\,|\,z\in L_\lambda \text{ and } \pi'(z)\in\boldsymbol{\phi}+W_\lambda\}. \tag{3}$$

$L_Q(\boldsymbol{\phi})$ is derived, alternatively, from $G_m(\boldsymbol{\phi})\equiv\pi(G\cap(\boldsymbol{\phi}+E'_2))=(G_{X'}-\phi_0)\cup\zeta(G_{X'}-\phi_1)\cup\ldots\cup\zeta^{m-1}(G_{X'}-\phi_{m-1})$ with the dual-grid method.

$L_Q(\boldsymbol{\phi})$ belongs to the same local-isomorphism class (LI class) as the original quasilattice L_Q does if $m=2^a3^b$ with a and b being integers, e.g. $m=2$, 3 and 4. This is because $2m$, the dimension of the starting lattice L, is equal in this case to the minimal dimension, $\varphi(n)$, of an n-gonal lattice. On the contrary, if $m\neq 2^a3^b$, e.g. $m=5$, $L_Q(\boldsymbol{\phi})$ may belong to a different LI class from that for L_Q. Moreover, the macroscopic point symmetry of $L_Q(\boldsymbol{\phi})$ can be lower than D_n for some $\boldsymbol{\phi}$. A classification of quasiperiodic patterns, $L_Q(\boldsymbol{\phi})$, will be performed by introducing an invariant(s) of $\boldsymbol{\phi}$ in a similar way as in the case of linear grids (Niizeki 1988b).

A multiple periodic grid $G_m(\boldsymbol{\phi})$ is called regular if vertices of any one of the periodic grids in $G_m(\boldsymbol{\phi})$ are located off others and there exist no multiple crossing points among different bonds of the periodic grids. $G_m(\boldsymbol{\phi})$ is regular for a generic $\boldsymbol{\phi}$. If $G_m=G_m(0)$ is regular, N_Q (or L_Q) as well as G_m has D_n as its exact point symmetry. However, the origin of G_m is a singular vertex for any m if $X'=\mathrm{T}$ or D. Moreover, the double-diced grid given in figure 4 has another kind of singular vertex; a six-pronged vertex of one periodic grid is located on a bond of the other. A singular grid is transformed to a regular one by an infinitesimal change of the phase vector $\boldsymbol{\phi}$. On this procedure, the exact symmetry of N_Q is broken 'spontaneously'. We can observe it in the structure in the central regular dodecagon in figure 3; the two hexagons in the dodecagon are dual to the six-pronged vertices of G_{D} and ζG_{D} which coincide at the origin. Figure 3 includes other structures breaking the exact dodecagonal symmetry.

The main part of constructing the quasiperiodic network $N_Q=\{L_Q, B_Q\}$ from the m-tuple periodic grid $G_m=G_{X'}\cup\zeta G_{X'}\cup\ldots\cup\zeta^{m-1}G_{X'}$ is a topological work, which is difficult to treat with a computer. Fortunately, there exists an algorithm for calculating the vertices of the tile which is dual to a cell of G_m. Let z be a representative point in the cell. Then, we can calculate the lattice points $z_0, z_1, \ldots, z_{m-1}\in L_X$ which satisfy

$z \in \zeta^k C(z_k)$, $k=0, 1, \ldots, m=1$, so that the dual lattice point of L_Q to the cell is given by $z_0+\zeta z_1+\ldots+\zeta^{m-1} z_{m-1}$. If a vertex of G_m is given, we can readily calculate representative points in the cells which adjoin at the vertex. Figure 3 (or figures 2 and 3 in Niizeki 1989a) has been drawn in this way with a computer.

If $m=2$ ($n=12$), the internal space as well as the external space is two-dimensional and we obtain $\pi'(z)=z_0-\zeta z_1$ because $\zeta_1=-\zeta$ ($\eta=-1$). It follows from this and the equality $\Gamma(z)=C(z_0)\times C(z_1)$ that $\pi'(\Gamma(z))=C(z_0)-\zeta C(z_1)=\{z-\zeta z' \mid z\in C(z_0)$ and $z'\in C(z_1)\}$. Using this result, we can easily obtain windows to be assigned to the sublattices for $X=$ T, HC, K or D. The windows obtained in this way for $X=$ HC and K have been presented already in Niizeki (1988a, 1989b), respectively.

We have shown that the tile statistics of a QPT obtained from a linear grid is easily calculated (Niizeki 1988b). This can be readily extended to the calculation of the tile statistics for the case of a multiple-honeycomb grid or a multiple-diced grid, which we will discuss in the appendix.

Before closing the paper, we present a remark: the reason why everything is simple for the quasilattice investigated in this paper derives from the fact that the starting lattice in the projection method is a direct product of equivalent 2D lattices. A decagonal lattice in 4D or a heptagonal lattice in 6D cannot be represented as a direct product of 2D lattices, so that a 'Bravais-type' decagonal quasilattice as presented in Niizeki (1989a) or a heptagonal one cannot be obtained from a multiple-periodic grid.

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Appendix

The tile statistics of a QPT are determined by the vertex statistics of the dual grid. The vertices of $G_m=G_{X'}\cup\zeta G_{X'}\cup\ldots\cup\zeta^{m-1}G_{X'}$ are divided into two groups V_1 and V_2 are presented in § 5. Since G_m is a superposition of periodic grids, the statistics of the vertices in V_1 are readily calculated. We consider V_2 in the case of $X'=$ D. G_D can be embedded into a triangular grid G_T, which is a linear grid; G_D is obtained from G_T if every grid line is cut by the ratio 1/3 so that it is changed into a periodic array of segments. The ratio of the period of the array to the distance between two grid lines in G_T is equal to $2/\sqrt{3}$, which is irrational. Therefore, we can conclude that the statistics of the vertices in V_2 are $(\frac{2}{3})^2$ times those in the case of G_T. Thus, the statistics of a triangle, a hexagon, a square and a rhombus in the tiling in figure 3 are proportional to 2: $1/2:1/\sqrt{3}:1/\sqrt{3}$. By a similar argument, we find that the statistics of a triangle, a square and a rhombus in a dodecagonal QPT obtained from the double-honeycomb grid (Stampfli 1986) are proportional to $2:1/\sqrt{3}:1/\sqrt{3}$.

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